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Remarks on Some Associated Laguerre Integral Results

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Abstract—Motivated essentially by their possible need in a fairly large number of physical and chemical contexts, Mavromatis and Alassar [1] derived several associated Laguerre integral results by eliminating an unnecessary constraint used in an earlier paper on the subject by Mavromatis [2]. The main object of the present sequel to these recent works is to investigate and apply much more general families of integral formulas, involving products of two or more Laguerre polynomials, which have been considered in the mathematical literature rather extensively.

Keywords—Laguerre polynomials, Orthogonality property, Generalized hypergeometric function, Noncentral Gamma distributions, Appell functions, Chu-Vandermonde theorem, Degrees of freedom, Coulomb and oscillator systems, Quantum numbers.

In the currently popular notations (see, for example, [3, Chapter V]), the Laguerre polynomials $L_n^{(\alpha)}(x)$ (of order α and degree n in x) are defined by

$$L_n^{(\alpha)}(x) := \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \frac{(-x)^k}{k!} \tag{1}$$

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or, equivalently, by

$$L_n^{(\alpha)}(x) = \binom{n+\alpha}{n} {}_1F_1(-n; \alpha+1; x), \tag{2}$$

where, as usual, ${}_pF_q$ denotes a generalized hypergeometric function with p numerator and q denominator parameters. These polynomials satisfy the following orthogonality property:

$$\begin{gathered}\int_0^\infty x^\alpha e^{-x} L_m^{(\alpha)}(x) L_n^{(\alpha)}(x)\,dx = \binom{n+\alpha}{n}\Gamma(\alpha+1)\delta_{m,n} \\ \left(\Re(\alpha) > -1;\ m,n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}\right),\end{gathered} \tag{3}$$

where $\delta_{m,n}$ is the Kronecker delta and $\mathbb{N}$ is the set of *positive* integers. Throughout this paper, we make use of the following *general* form of the relatively more familiar binomial coefficients:

$$\binom{\lambda}{\mu} := \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\mu+1)\Gamma(\mu+1)} = \binom{\lambda}{\lambda-\mu} \qquad (\lambda, \mu \in \mathbb{C}),$$

so that, obviously, we have

$$\binom{\lambda}{0} = 1 \quad \text{and} \quad \binom{\lambda}{n} = \frac{\lambda(\lambda-1)\cdots(\lambda-n+1)}{n!} \qquad (n \in \mathbb{N}).$$

Making use of a slightly different notation $\mathcal{L}_n^\alpha(x)$ for the so-called *associated* Laguerre polynomials, where

$$\mathcal{L}_n^\alpha(x) = \Gamma(\alpha+n+1) L_n^{(\alpha)}(x),$$

Mavromatis [2] evaluated the following integral involving the product of two Laguerre polynomials:

$$\begin{gathered}\int_0^\infty x^\mu e^{-x} L_m^{(\alpha)}(x) L_n^{(\beta)}(x)\,dx = \binom{m+\alpha}{m}\binom{n+\beta-\mu-1}{n}\Gamma(\mu+1) \\ \cdot {}_3F_2(-m, \mu+1, \mu-\beta+1; \alpha+1, \mu-\beta-n+1; 1) \\ \left(\Re(\mu) > -1;\ m,n \in \mathbb{N}_0\right),\end{gathered} \tag{4}$$

where (*and elsewhere in this paper*) it is tacitly assumed that the various parameters involved are so constrained that no zeros appear in the denominator on the right-hand side.

Subsequently, Lee [4] deduced the integral formula (4) as an easy consequence of a well-known (*rather classical*) result involving the product of several Laguerre polynomials (see, for example, [5, p. 260, Problem 2(ii)]; see also [6] for a probabilistic derivation of this general result, which is based mainly upon the moments of certain noncentral Gamma distributions). On the other hand, Lee *et al.* [7] considered a family of such integrals as (for example) in (4) involving products of Laguerre, Hermite, and other classical orthogonal polynomials, pointed out relevant connections of some of these integral formulas with various known integrals, and also investigated the computational and numerical aspects of the results presented by them (see, for details, [7]).

In view of the potential for their usefulness in various physical and chemical contexts, Mavromatis and Alassar [1] deduced many *further* special cases of the integral formula (4) *without* an obviously unnecessary parametric restriction by which Mavromatis [2] had earlier constrained (4). Here, in the present sequel to each of these recent works, we begin by recalling the following special case of the aforementioned general (classical) result:

$$\begin{gathered}\int_0^\infty x^{\rho-1} e^{-\sigma x} L_m^{(\alpha)}(\lambda x) L_n^{(\beta)}(\mu x)\,dx \\ = \binom{m+\alpha}{m}\binom{n+\beta}{n}\frac{\Gamma(\rho)}{\sigma^\rho} F_2\left[\rho, -m, -n; \alpha+1, \beta+1; \frac{\lambda}{\sigma}, \frac{\mu}{\sigma}\right] \\ \left(\Re(\rho) > 0;\ \Re(\sigma) > 0;\ m,n \in \mathbb{N}_0\right),\end{gathered} \tag{5}$$

where F_2 denotes one of the four Appell functions defined by (cf., e.g., [5, p. 53, equation 1.6 (5)])

$$\begin{aligned} F_2\left[a,b,b';c,c';x,y\right] &:= \sum_{r,s=0}^{\infty} \frac{(a)_{r+s}(b)_r\,(b')_s}{(c)_r\,(c')_s}\,\frac{x^r}{r!}\,\frac{y^s}{s!} \\ &= \sum_{r=0}^{\infty} \frac{(a)_r(b)_r}{(c)_r}\,{}_2F_1\left(a+r,b';c';y\right)\frac{x^r}{r!} \\ &\quad \left(|x|+|y|<1;\ c,c' \notin \mathbb{Z}_0^- := \{0,-1,-2,\dots\}\right), \end{aligned} \tag{6}$$

$(a)_r := \Gamma(a+r)/\Gamma(a)$ being the Pochhammer symbol or the *shifted* factorial, since $(1)_r = r!$ $(r \in \mathbb{N}_0)$.

In view of the familiar Chu-Vandermonde theorem [5, p. 30, equation 1.2 (8)],

$${}_2F_1(-n,b;c;1) = \frac{(c-b)_n}{(c)_n} \qquad \left(n \in \mathbb{N}_0;\ c \notin \mathbb{Z}_0^-\right), \tag{7}$$

a special case of (5) when $\mu = \sigma$ readily yields the integral formula

$$\begin{aligned} \int_0^\infty x^{\rho-1}\,e^{-\sigma x}\,L_m^{(\alpha)}(\lambda x)L_n^{(\beta)}(\sigma x)\,dx &= \binom{m+\alpha}{m}\binom{n+\beta-\rho}{n}\frac{\Gamma(\rho)}{\sigma^\rho} \\ &\cdot {}_3F_2\left(-m,\rho,\rho-\beta;\ \alpha+1,\rho-\beta-n;\ \frac{\lambda}{\sigma}\right) \\ &\quad (\Re(\rho)>0;\ \Re(\sigma)>0;\ m,n \in \mathbb{N}_0), \end{aligned} \tag{8}$$

which is a known result recorded by (for example) Prudnikov *et al.* [8, p. 478, Entry 2.19.14.8], who indeed also gave a *further* special case of (8) [8, p. 478, Entry 2.19.14.15] when

$$\rho = \mu + 1 \quad \text{and} \quad \lambda = \sigma (= 1).$$

As a matter of fact, this familiar *further* special case of the known result (8) is precisely the integral formula (4) which was proven, almost three decades *before* Mavromatis [2], by Carlitz [9, p. 339, equation (17)]. See also the work of Srivastava [10, p. 211] in which the integral formula (4) was applied with a view to evaluating a certain integral involving the product of two Bessel polynomials.

Carlitz [9, p. 340] also gave the following symmetrical form of the integral formula (4) above (see also [7, p. 309, equation (15)]):

$$\begin{aligned} \int_0^\infty x^\mu\,e^{-x}\,L_m^{(\alpha)}(x)L_n^{(\beta)}(x)\,dx &= \binom{m+\alpha-\mu-1}{m}\binom{n+\beta-\mu-1}{n}\Gamma(\mu+1) \\ &\cdot {}_3F_2(-m,-n,\mu+1;\mu-\alpha-m+1,\mu-\beta-n+1;1) \\ &\quad (\Re(\mu)>-1;\ m,n \in \mathbb{N}_0). \end{aligned} \tag{9}$$

Moreover, the following combinatorial series representation for the integral in (4) and (9) was given by Rassias and Srivastava [11, p. 171, equation (19)]:

$$\begin{aligned} &\int_0^\infty x^\mu\,e^{-x}\,L_m^{(\alpha)}(x)L_n^{(\beta)}(x)\,dx \\ &= (-1)^{m+n}\Gamma(\mu+1)\sum_{k=0}^{\min(m,n)}\binom{\mu-\alpha}{m-k}\binom{\mu-\beta}{n-k}\binom{k+\mu}{k} \\ &\quad (\Re(\mu)>-1;\ m,n \in \mathbb{N}_0), \end{aligned} \tag{10}$$

which (for $\mu = \alpha = \beta$) leads us immediately to the orthogonality property (3).

We now turn to a known multiplication formula for the Laguerre polynomials in the form [5, p. 261, Problem 3(iii)]

$$L_n^{(\alpha)}(xy) = \sum_{k=0}^{n} \binom{n+\alpha}{n-k} y^k L_k^{(\beta)}(x)\, {}_2F_1(-n+k, \beta+k+1; \alpha+k+1; y). \tag{11}$$

In light of the orthogonality property (3), by rewriting (11) in its *equivalent* forms

$$L_m^{(\alpha)}(\lambda x) = \sum_{j=0}^{m} \binom{m+\alpha}{m-j} \left(\frac{\lambda}{\sigma}\right)^j L_j^{(\gamma)}(\sigma x)\, {}_2F_1\left(-m+j, \gamma+j+1; \alpha+j+1; \frac{\lambda}{\sigma}\right) \tag{12}$$

and

$$L_n^{(\beta)}(\mu x) = \sum_{k=0}^{n} \binom{n+\beta}{n-k} \left(\frac{\mu}{\sigma}\right)^k L_k^{(\gamma)}(\sigma x)\, {}_2F_1\left(-n+k, \gamma+k+1; \beta+k+1; \frac{\mu}{\sigma}\right), \tag{13}$$

it is not difficult to deduce the following *alternative* form of the known result (5) with, of course, $\rho = \gamma + 1$:

$$\begin{aligned} &\int_0^\infty x^\gamma e^{-\sigma x} L_m^{(\alpha)}(\lambda x) L_n^{(\beta)}(\mu x)\, dx \\ &= \frac{\Gamma(\gamma+1)}{\sigma^{\gamma+1}} \sum_{k=0}^{\min(m,n)} \binom{m+\alpha}{m-k}\binom{n+\beta}{n-k}\binom{k+\gamma}{k}\left(\frac{\lambda\mu}{\sigma^2}\right)^k \\ &\cdot {}_2F_1\left(-m+k, \gamma+k+1; \alpha+k+1; \frac{\lambda}{\sigma}\right) {}_2F_1\left(-n+k, \gamma+k+1; \beta+k+1; \frac{\mu}{\sigma}\right) \\ &(\Re(\gamma) > -1;\ \Re(\sigma) > 0;\ m, n \in \mathbb{N}_0). \end{aligned} \tag{14}$$

At least two special cases of the integral formula (14) are worthy of note. First of all, by means of the Chu-Vandermonde theorem (7), a special case of (14) when $\mu = \sigma$ yields (cf. equation (8) above)

$$\begin{aligned} &\int_0^\infty x^\gamma e^{-\sigma x} L_m^{(\alpha)}(\lambda x) L_n^{(\beta)}(\sigma x)\, dx \\ &= \frac{\Gamma(\gamma+1)}{\sigma^{\gamma+1}} \sum_{k=0}^{\min(m,n)} \binom{m+\alpha}{m-k}\binom{k+\gamma}{k}\frac{(\beta-\gamma)_{n-k}}{(n-k)!}\left(\frac{\lambda}{\sigma}\right)^k \\ &\cdot {}_2F_1\left(-m+k, \gamma+k+1; \alpha+k+1; \frac{\lambda}{\sigma}\right) \\ &(\Re(\gamma) > -1;\ \Re(\sigma) > 0;\ m, n \in \mathbb{N}_0), \end{aligned} \tag{15}$$

which, for $\gamma = \beta$, reduces at once to the integral formula

$$\begin{aligned} &\int_0^\infty x^\beta e^{-\sigma x} L_m^{(\alpha)}(\lambda x) L_n^{(\beta)}(\sigma x)\, dx \\ &= \binom{m+\alpha}{m-n}\binom{n+\beta}{n}\frac{\lambda^n \Gamma(\beta+1)}{\sigma^{\beta+n+1}}\, {}_2F_1\left(-m+n, \beta+n+1; \alpha+n+1; \frac{\lambda}{\sigma}\right) \\ &(\Re(\beta) > -1;\ \Re(\sigma) > 0;\ m \geqq n \geqq 0\ (m, n \in \mathbb{N}_0)). \end{aligned} \tag{16}$$

Second, by setting $\lambda = \mu = \sigma$ in (14), we similarly obtain the integral formula

$$\begin{aligned} &\int_0^\infty x^\gamma e^{-\sigma x} L_m^{(\alpha)}(\sigma x) L_n^{(\beta)}(\sigma x)\, dx \\ &= \frac{\Gamma(\gamma+1)}{\sigma^{\gamma+1}} \sum_{k=0}^{\min(m,n)} \binom{k+\gamma}{k}\frac{(\alpha-\gamma)_{m-k}}{(m-k)!}\,\frac{(\beta-\gamma)_{n-k}}{(n-k)!} \\ &(\Re(\gamma) > -1;\ \Re(\sigma) > 0;\ m, n \in \mathbb{N}_0), \end{aligned} \tag{17}$$

which, for $\gamma = \beta$, immediately yields

$$\int_0^\infty x^\beta \, e^{-\sigma x} \, L_m^{(\alpha)}(\sigma x) L_n^{(\beta)}(\sigma x) \, dx = \frac{\Gamma(\beta+1)}{\sigma^{\beta+1}} \binom{n+\beta}{n} \frac{(\alpha-\beta)_{m-n}}{(m-n)!} \tag{18}$$
$$(\Re(\beta) > -1; \ \Re(\sigma) > 0; \ m \geqq n \geqq 0 \ (m, n \in \mathbb{N}_0)).$$

A *further* special case of (16) when $\beta = \alpha$ can be written in the form

$$\int_0^\infty x^\alpha \, e^{-\sigma x} \, L_m^{(\alpha)}(\lambda x) L_n^{(\alpha)}(\sigma x) \, dx = \frac{\Gamma(\alpha+m+1)}{\sigma^{\alpha+m+1}} \frac{(\sigma-\lambda)^{m-n}}{(m-n)!} \frac{\lambda^n}{n!} \tag{19}$$
$$(\Re(\alpha) > -1; \ \Re(\sigma) > 0; \ m \geqq n \geqq 0 \ (m, n \in \mathbb{N}_0)),$$

which corresponds to the corrected version of an integral formula recorded by Prudnikov *et al.* [8, p. 478, Entry 2.19.14.10].

The integral formula (10) is precisely the same as (17) *with* $\sigma = 1$. Its special case when

$$\mu - \alpha - m \in \mathbb{N}_0 \qquad \text{and} \qquad \mu - \beta - n \in \mathbb{N}_0$$

was given by Morse and Feshbach [12, p. 785].

Finally, in the *equivalent* integral formulas (10) and (17), we set

$$\mu = \gamma = \beta + n - m + l \qquad (n \geqq m - l \geqq 0; \ l, m, n \in \mathbb{N}_0)$$

and we readily obtain

$$\int_0^\infty x^{\beta+n-m+l} \, e^{-\sigma x} \, L_m^{(\alpha)}(\sigma x) L_n^{(\beta)}(\sigma x) \, dx$$
$$= \frac{(-1)^{m+n}}{\sigma^{\beta+n-m+l+1}} \sum_{k=m-l}^{m} \binom{\beta-\alpha+n-m+l}{m-k} \binom{n-m+l}{k-m+l} \frac{\Gamma(\beta+n-m+l+k+1)}{k!} \tag{20}$$
$$(\Re(\beta) > -1; \ \Re(\sigma) > 0; \ n \geqq m - l \geqq 0 \ (l, m, n \in \mathbb{N}_0)).$$

The *main* results of Mavromatis and Alassar [1, p. 904, equations (5) and (10)] happen to correspond to the special cases of (20) (*with* $\sigma = 1$) when

$$l = 0 \qquad \text{and} \qquad l = 1,$$

respectively. The corresponding special cases ($\sigma = 1$; $l = 0, 1$) of the following obvious consequence of (20) when $n = m - l \geqq 0$ $(l, m, n \in \mathbb{N}_0)$:

$$\int_0^\infty x^\beta \, e^{-\sigma x} \, L_m^{(\alpha)}(\sigma x) L_{m-l}^{(\beta)}(\sigma x) \, dx$$
$$= (-1)^l \frac{\Gamma(\beta+1)}{\sigma^{\beta+1}} \binom{\beta+m-l}{m-l} \binom{\beta-\alpha}{l} \tag{21}$$
$$(\Re(\beta) > -1; \ \Re(\sigma) > 0; \ m - l \geqq 0 \ (l, m \in \mathbb{N}_0))$$

were also presented by Mavromatis and Alassar [1, p. 904, equation (6); p. 905, equation (11)].

Numerous further (known or new) consequences of these last integral formulas (20) and (21) can be deduced fairly easily. The details involved in these derivations are being left as an exercise for the interested reader.

The parameter σ in the integral formula (14), and also in its consequences such as (15)–(21), does not really offer an *extra* degree of freedom. Indeed, *without* any loss of generality, by appropriately setting

$$x \longmapsto \frac{x}{\sigma}, \qquad \lambda \longmapsto \lambda\sigma, \qquad \text{and} \qquad \mu \longmapsto \mu\sigma,$$

each of these integral formulas can be recast in a form *independent* of σ. Nonetheless, the general result (14) (with *at least* two degrees of freedom introduced by the parameters λ and μ) is very useful in physical contexts. Among other places, such general integral formulas as (14) arise when we obtain analytic matrix elements of moments between different states for Coulomb systems in which, unlike oscillator systems, the arguments of the Laguerre polynomials are different for states with different principal quantum numbers.

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